On the hom-associative Weyl algebras

Noncommutative Rings and their Applications VII, Lens (via Zoom), 2021

Per Bäck, <per.back@mdh.se>

July 6, 2021

Mälardalen University, Sweden

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Motivation

Motivation

Many Lie algebras are *rigid*; they cannot be deformed without altering the Jacobi identity (e.g. any semisimple Lie algebra in characteristic zero is rigid). Remedy: generalize Lie algebras into *hom-Lie algebras*, as introduced in [HLS06]. In this context, *hom-associative algebras* arise naturally.

Another remedy: deform the universal enveloping algebra $U(L)$ of the Lie algebra L. But $U(L)$ can also be rigid as an associative algebra (L is *strongly rigid*). However, $U(L)$ need not be rigid as a hom-associative algebra.

[HLS06] J.T. Hartwig, D. Larsson, and S.D. Silvestrov. "Deformations of Lie algebras using σ-derivations". In: *J. Algebra* 295.2 (2006).

Motivation

Non-commutative polynomial rings – or *Ore extensions* – were introduced by Ore [Ore33], and recently generalized to the hom-associative setting [BRS18].

Ore extensions include many rigid algebras, e.g. rigid universal enveloping algebras of Lie algebras, and the Weyl algebras in characteristic zero. However, these can often be deformed as hom-associative Ore extensions.

This talk is about deformed Weyl algebras – the *hom-associative Weyl algebras* [BR20a; BR20b] – and a deformed Dixmier conjecture [Dix68].

[Ore33] O. Ore. "Theory of Non-Commutative Polynomials". In: *Ann. Math.* 34.3 (1933).

[BRS18] P. Bäck, J. Richter, and S. Silvestrov. "Hom-associative Ore extensions and weak unitalizations". In: *Int. Electron. J. Algebra* 24 (2018).

[BR20a; BR20b] P. Bäck and J. Richter. "On the hom-associative Weyl algebras". In: *J. Pure Appl. Algebra* 224.9 (2020); P. Bäck and J. Richter. "The hom-associative Weyl algebras in prime characteristic". In: arXiv:2012.11659 (2020).

[Dix68] J. Dixmier. "Sur les algèbres de Weyl". In: *Bull. Soc. Math. France* 96 (1968).

Hom-algebras

Hom-associative algebras: preliminaries

Definition (Hom-everything)

A *hom-associative algebra* over an associative, commutative, and unital ring R, is a triple $(M, \cdot, \alpha)$ consisting of an R-module M, an R-bilinear map $\cdot\colon M \times M \to M$, and an R-linear map $\alpha\colon M \to M$, satisfying,

$$\alpha(a) \cdot (b \cdot c) = (a \cdot b) \cdot \alpha(c), \quad \forall a, b, c \in M.$$

A *hom-associative ring* is a hom-associative algebra over $\mathbb{Z}$.

A map $f\colon A \to B$ between hom-associative algebras is a *homomorphism* if it is linear, multiplicative, and $f \circ \alpha_A = \alpha_B \circ f$.

A left (right) ideal I s.t. $\alpha(I) \subseteq I$ is a left (right) *hom-ideal*.

A hom-associative algebra A is called *weakly unital* with *weak identity* $e \in A$ if for all $a \in A$, $e \cdot a = a \cdot e = \alpha(a)$.

Proposition ([BRS18])

Any multiplicative hom-associative algebra can be embedded as a hom-ideal into a multiplicative, weakly unital hom-associative algebra.

Proposition ([Yau09])

Let A be a unital, associative algebra with identity 1_A, α an algebra endomorphism on A, and define $*\colon A\times A\to A$ for all $a,b\in A$ by

$$a*b:=\alpha(a\cdot b).$$

Then $(A,*,\alpha)$ is a weakly unital hom-associative algebra with weak identity 1_A.

Definition (Multi-parameter formal hom-associative deformation)

An *n-parameter formal deformation* of a hom-associative algebra $(M,\cdot_0,\alpha_0)$ over R, is a hom-associative algebra $(M[\![t_1,\ldots,t_n]\!],\cdot_t,\alpha_t)$ over $R[\![t_1,\ldots,t_n]\!]$, where

$$\cdot_t=\sum_{i\in\mathbb{N}^n}\cdot_i t^i,\quad \alpha_t=\sum_{i\in\mathbb{N}^n}\alpha_i t^i.$$

Here, $i:=(i_1,\ldots,i_n)$, $t:=(t_1,\ldots,t_n)$, and $t^i:=t_1^{i_1}\cdots t_n^{i_n}$.

[Yau09] D. Yau. "Hom-algebras and Homology". In: *J. Lie Theory* 19.2 (2009).

Definition (Hom-Lie algebra)

A *hom-Lie algebra* over an associative, commutative, and unital ring R is a triple $(M, [\cdot,\cdot], \alpha)$ where M is an R-module, $\alpha\colon M \to M$ a linear map, and $[\cdot,\cdot]\colon M \times M \to M$ a bilinear and alternative map, satisfying:

$$[\alpha(a), [b, c]] + [\alpha(c), [a, b]] + [\alpha(b), [c, a]] = 0, \quad \forall a, b, c \in M.$$

Proposition ([MS08])

Let $(M, \cdot, \alpha)$ be a hom-associative algebra with commutator $[\cdot,\cdot]$. Then $(M, [\cdot,\cdot], \alpha)$ is a hom-Lie algebra.

Definition (Multi-parameter formal hom-Lie deformation)

An *n-parameter formal deformation* of a hom-Lie algebra $(M, [\cdot,\cdot]_0, \alpha_0)$ over R, is a hom-Lie algebra $(M[\![t_1,\ldots,t_n]\!], [\cdot,\cdot]_t, \alpha_t)$ over $R[\![t_1,\ldots,t_n]\!]$, where

$$[\cdot,\cdot]_t = \sum_{i\in\mathbb{N}^n} [\cdot,\cdot]_i t^i, \quad \alpha_t = \sum_{i\in\mathbb{N}^n} \alpha_i t^i.$$

Here, $i := (i_1, \ldots, i_n)$, $t := (t_1, \ldots, t_n)$, and $t^i := t_1^{i_1} \cdots t_n^{i_n}$.

[MS08] A. Makhlouf and S.D. Silvestrov. "Hom-algebra structures". In: *J. Gen. Lie Theory Appl.* 2.2 (2008).

Non-commutative, associative polynomial rings

Associative Ore extensions: motivation

Let R be an associative and unital ring, and consider $R[x]$ as an additive group. Want to make this an associative, non-commutative, unital ring S:

$\deg(p \cdot q) \leq \deg(p) + \deg(q)$ for any $p, q \in S$,

$x^m \cdot x^n = x^{m+n}$ for any $m, n \in \mathbb{N}$,

For any $r \in R$, we need $x \cdot r = \sigma(r)x + \delta(r)$ for some $\sigma, \delta \colon R \to R$ (while S is a left R-module). Iterating, we get

$$rx^m \cdot sx^n = \sum_{i \in \mathbb{N}} (r\pi_i^m(s))x^{i+n},$$

where $\pi_i^m \colon R \to R$ is the sum of all $\binom{m}{i}$ compositions of i copies of σ and $m - i$ copies of δ. For example, $\pi_1^2(r) = \sigma(\delta(r)) + \delta(\sigma(r))$.

Associative Ore extensions: σ and δ

S should be an associative and unital ring, so for any $r, s \in R$,

$$x \cdot (r+s) = x \cdot r + x \cdot s \quad \text{(left distributivity)},$$
$$x \cdot (rs) = (x \cdot r) \cdot s \quad \text{(associativity)},$$
$$x \cdot 1_R = 1_R \cdot x = x \quad \text{(unitality)}.$$

This implies

$$\sigma(1_R) = 1_R,$$
$$\sigma(r+s) = \sigma(r) + \sigma(s),$$
$$\sigma(rs) = \sigma(r)\sigma(s),$$

so σ needs to be an *endomorphism*. Moreover,

$$\delta(r+s) = \delta(r) + \delta(s),$$
$$\delta(rs) = \sigma(r)\delta(s) + \delta(r)s,$$

so δ is a *σ-derivation* (if $\sigma = \mathrm{id}_R$, a *derivation*). For such σ and δ we get an associative and unital ring $R[x; \sigma, \delta]$, the *Ore extension of R*.

Associative Ore extensions: examples

Let R be an associative and unital ring, and $r \in R$.

Example (Polynomial ring)

A *polynomial ring* over R, written $R[x]$, is $R[x; \mathrm{id}_R, 0_R]$.
Here, $x \cdot r = rx$.

Example (Skew-polynomial ring)

A *skew-polynomial ring* over R is $R[x; \sigma, 0_R]$ for some endomorphism σ.
Here, $x \cdot r = \sigma(r)x$.

Example (Differential polynomial ring)

A *differential polynomial ring* over R is $R[x; \mathrm{id}_R, \delta]$, δ a derivation.
Here, $x \cdot r = rx + \delta(r)$.

The *Weyl algebra* A_1 over a field K, is $K\langle x, y\rangle / (x \cdot y - y \cdot x - 1_K)$.
$A_1 = K[y][x; \mathrm{id}_{K[y]}, \mathrm{d}/\mathrm{d}y]$.

Non-commutative, hom-associative polynomial rings

Definition (Non-associative, non-unital Ore extension)

If R is a non-associative, non-unital ring, a map $\beta\colon R \to R$ is *left R-additive* if for all $r, s, t \in R$, $r \cdot \beta(s+t) = r \cdot (\beta(s) + \beta(t))$.

A *non-associative, non-unital Ore extension* of R, $R[x;\sigma,\delta]$, where σ and δ are left R-additive maps on R, is the additive group $R[x]$ with

$$rx^m \cdot sx^n := \sum_{i\in\mathbb{N}} (r\pi_i^m(s))\, x^{i+n}, \quad \forall r, s \in R.$$

Theorem (Hilbert's basis theorem for non-associative rings [BR18])

Let R be a non-associative, unital ring, σ an automorphism and δ a σ-derivation on R. If R is right (left) noetherian, then so is $R[x;\sigma,\delta]$.

[BR18] P. Bäck and J. Richter. "Hilbert's basis theorem for non-associative and hom-associative Ore extensions". In: arXiv:1804.11304 (2018).

Hom-associative Ore extensions: sufficiency

We extend any map α on R *homogeneously* to $R[x;\sigma,\delta]$ by $\alpha(\sum_{i\in\mathbb{N}} r_i x^i) := \sum_{i\in\mathbb{N}} \alpha(r_i)x^i$, $r_i \in R$.

Proposition ([BRS18])

Let R be a hom-associative ring with twisting map α, σ an endomorphism and δ a σ-derivation that both commute with α. Then $R[x;\sigma,\delta]$ is a hom-associative Ore extension, α extended homogeneously to $R[x;\sigma,\delta]$.

Proposition ([BRS18])

Let R be a unital, associative ring, σ an endomorphism, δ a σ-derivation, and α an endomorphism that commutes with σ and δ. Then $(R[x;\sigma,\delta],*,\alpha)$ is a weakly unital, hom-associative Ore extension, α extended homogeneously to $R[x;\sigma,\delta]$.

The above conditions turn out to be *almost* necessary as well.

The hom-associative Weyl algebras

Lemma ([BRS18], [BR20b])

Let K be a field and α an endomorphism on $K[y]$. Then α commutes with $\mathrm{d}/\mathrm{d}y$ if and only if

$$\alpha(y) = \begin{cases} k_0 + y & \textit{if } \operatorname{char}(K) = 0, \\ k_0 + y + k_p y^p + k_{2p} y^{2p} + \ldots & \textit{if } \operatorname{char}(K) = p > 0. \end{cases}$$

Here, $k_0, k_p, k_{2p}, \ldots \in K$.

Rename the above map α_k, $k := \begin{cases} k_0 & \text{if } \operatorname{char}(K) = 0, \\ (k_0, k_p, k_{2p}, \ldots) & \text{if } \operatorname{char}(K) = p > 0. \end{cases}$

Definition (The hom-associative Weyl algebras [BRS18], [BR20b])

The *hom-associative Weyl algebras* A_1^k are $(A_1, *, \alpha_k)$ where α_k is extended homogeneously to $A_1 = K[y][x; \mathrm{id}_{K[y]}, \mathrm{d}/\mathrm{d}y]$.

If $k = 0$, then $\alpha_k = \mathrm{id}_{A_1}$, so $A_1^0 = A_1$.

Proposition ([BR20a], [BR20b])

1_K is a unique weak identity in A_1^k.

A_1^k contains no zero divisors.

A_1^k is simple if and only if $\operatorname{char}(K) = 0$.

A_1^k is power associative if and only if $k = 0$.

$$N(A_1^k) = \begin{cases} A_1^k & \text{if } k = 0, \\ \{0\} & \text{otherwise.} \end{cases}$$

$$C(A_1^k) = C(A_1) = \begin{cases} K & \text{if } \operatorname{char}(K) = 0, \\ K[x^p, y^p] & \text{if } \operatorname{char}(K) = p > 0. \end{cases}$$

$$\begin{aligned} \operatorname{Der}_K(A_1^k) &\subseteq \operatorname{Der}_K(A_1) \\ &= \begin{cases} \operatorname{InnDer}_K(A_1) & \text{if } \operatorname{char}(K) = 0, \\ C(A_1)E_x \oplus C(A_1)E_y \oplus \operatorname{InnDer}_K(A_1) & \text{if } \operatorname{char}(K) = p > 0. \end{cases} \end{aligned}$$

$E_x, E_y \in \operatorname{Der}_K(A_1)$, $E_x(x) = y^{p-1}$, $E_x(y) = 0$, $E_y(x) = 0$, $E_y(y) = x^{p-1}$.

The hom-associative Weyl algebras

Proposition ([BR20a], [BR20b])

$A_1^k \cong A_1^\ell$ *if* $k, \ell \neq 0$ *and* $\mathrm{char}(K) = 0$.

$A_1^k \cong A_1^\ell$ *does not hold in general if* $k, \ell \neq 0$ *and* $\mathrm{char}(K) > 0$.

Proposition ([BR20a], [BR20b])

Every nonzero endomorphism on A_1^k *is injective.*

Every nonzero endomorphism on A_1^k *is surjective if* $k \neq 0$ *and* $\mathrm{char}(K) = 0$.

Not every nonzero endomorphism on A_1^k *is surjective if* $\mathrm{char}(K) > 0$.

Conjecture ([Dix68])

Every nonzero endomorphism on A_1 *is surjective if* $\mathrm{char}(K) = 0$.

Proposition ([BR20a], [BR20b])

A_1^k *is a multi-parameter formal deformation of* A_1.

The hom-Lie algebra of A_1^k *is a multi-parameter formal deformation of the Lie algebra of* A_1*, using the commutator as bracket.*

Thank you!